

Homework 7 Solution Key

Math 121A, Fall 2025.

Unless otherwise stated, all solutions require clear, well-written proofs.

Problem 1. (*Return of the Laplacian*) Taking partial derivatives and using the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0,$$

where in the last equality we used that f is twice continuously differentiable and so we may commute partial derivatives. An almost identical calculation similarly shows that v is also harmonic. [As a note](#), we call v the *harmonic conjugate* of u .

Problem 2. (a) $|z|$ is real, so $\operatorname{Re}(|z|) = |z|$ and $\operatorname{Im}(|z|) = 0$. We will show it is not holomorphic. Similar to what we did in class for $|z|^2$, we can compute the limits defining $f'(1)$ in two directions and find that they don't agree. We can also use the Cauchy-Riemann equations: if $u(z) = \operatorname{Re}(|z|)$ and $v(z) = \operatorname{Im}(|z|)$, then

$$\frac{\partial v}{\partial x} = 0 \quad \text{but} \quad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \neq 0.$$

(b) Since $\overline{x + iy} = x - iy$, we have $\operatorname{Re}(\bar{z}) = \operatorname{Re}(z)$ and $\operatorname{Im}(\bar{z}) = -\operatorname{Im}(z)$. This function is not holomorphic, which can be seen either by computing two limits (one in the real direction and another in the imaginary direction) for the derivative at 1 and seeing they aren't equal, or by checking the Cauchy-Riemann equations: if $u(z) = \operatorname{Re}(\bar{z})$ and $v(z) = \operatorname{Im}(\bar{z})$, then

$$\frac{\partial u}{\partial x} = 1, \quad \text{but} \quad \frac{\partial v}{\partial y} = -1.$$

(c) Let us compute, using the complex definition of sine from Euler's formula:

$$\begin{aligned} \sin(x + iy) &= \frac{1}{2i}(e^{i(x+iy)} - e^{-i(x+iy)}) = \frac{1}{2i}(e^{ix}e^{-y} - e^{-ix}e^y) \\ &= \frac{1}{2i}((\cos x + i \sin x)e^{-y} - (\cos x - i \sin x)e^y) \\ &= \frac{1}{2}((-i \cos x + \sin x)e^{-y} - (-i \cos x - \sin x)e^y) \\ &= \frac{1}{2}(\sin x(e^{-y} + e^y) + i \cos x(e^y - e^{-y})) \end{aligned}$$

So the real part is $\frac{1}{2} \sin x(e^y + e^{-y})$ and the imaginary part is $\frac{1}{2} \cos x(e^y - e^{-y})$.

This function is holomorphic since $\frac{d}{dz}(\sin z) = \cos z$.

(d) Let us compute:

$$\frac{1}{x+iy} = \frac{x-iy}{(x+iy)\overline{x+iy}} = \frac{x}{|x+iy|^2} - i \frac{y}{|x+iy|^2}.$$

So the real part is $\frac{x}{|x+iy|^2}$ and the imaginary part is $\frac{-y}{|x+iy|^2}$. This function is holomorphic since $\frac{d}{dz}(\frac{1}{z}) = -\frac{1}{z^2}$.

Problem 3. Problem 14.2.28: This is not a fully rigorous proof (the two potential problems are that it could be that $g(z) - g(z_0) = 0$, and that my “splitting” of the limit to evaluate $g'(z_0)$ was troublesome. To see a proper proof which circumvents these subtleties, see e.g. <https://web.williams.edu/Mathematics/lg5/A37W12/Chain.pdf>. While this proof is for real derivatives, the proof is identical.)

$$\begin{aligned} (f \circ g)'(z_0) &= \lim_{z \rightarrow z_0} \frac{(f(g(z)) - (f(g(z_0))))}{z - z_0} = \lim_{z \rightarrow z_0} \frac{(f(g(z)) - (f(g(z_0))))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0} \\ &= \lim_{g(z+h) \rightarrow g(h)} \frac{(f(g(z)) - (f(g(z_0))))}{g(z) - g(z_0)} g'(z_0) \\ &= f'(g(z_0))g'(z_0). \end{aligned}$$

Problem 4. (a) Problem 14.3.3

i. This rectangular integral may be broken into 4 pieces:

$$\begin{aligned} \oint_C z^2 dz &= \int_{-1}^1 (x+0i)^2 dx + \int_0^1 (1+yi)^2 idy + \int_1^{-1} (x+i)^2 dx + \int_1^0 (-1+yi)^2 idy \\ &= \int_{-1}^1 x^2 dx + \int_0^1 (1+2yi-y^2) idy - \int_{-1}^1 x^2 + 2xi - 1 dx - \int_0^1 (1-2yi-y^2) idy \\ &= \int_{-1}^1 -2xi + 1 dx + \int_0^1 -4y dy \\ &= 0. \end{aligned}$$

ii. This semicircular integral may be broken into 2 pieces:

$$\begin{aligned} \oint_C z^2 dz &= \int_{-1}^1 (x+0i)^2 dx + \int_0^\pi (e^{it})^2 (ie^{it}) dt \\ &= \frac{2}{3} + \int_0^\pi ie^{i3t} dt \\ &= \frac{2}{3} + \frac{1}{3}(e^{i3\pi} - 1) \\ &= 0. \end{aligned}$$

As expected, this integral is path-independent since z^2 is holomorphic.

(b) Problem 14.3.6

i. This quarter circle contour may be expressed by $\gamma(t) = e^{it}$, $t \in [0, \pi/2]$, so the integral is

$$\int_\gamma z dz = \int_0^{\pi/2} e^{it} i e^{it} dt = i \int_0^{\pi/2} e^{2it} dt = i \left(\frac{1}{2i} (e^{2i(\pi/2)} - 1) \right) = -1.$$

ii. This integral may be broken into two pieces:

$$\begin{aligned}\int_0^1 (1+iy)idy + \int_1^0 (x+i)dx &= \int_0^1 (1+iy)idy - \int_0^1 (x+i)dx \\ &= i - \left(\frac{1^2}{2} - \frac{0^2}{2}\right) - i - \left(\frac{1^2}{2} - \frac{0^2}{2}\right) = -1.\end{aligned}$$

As expected, this integral is path-independent since z is holomorphic.

(c) Problem 14.3.12

i. This straight line segment may be parameterized by $\gamma(t) = t(1+2i)$, $t \in [0, 1]$, so

$$\int_{\gamma} |z|^2 dz = \int_0^1 |t(1+2i)|^2 (1+2i) dt = 5(1+2i) \int_0^1 t^2 dt = \frac{5(1+2i)}{2}.$$

This rectangular contour may be broken into two pieces:

$$\int_{\gamma} |z|^2 dz = \int_0^2 |iy|^2 i dy + \int_0^1 |x+2i|^2 dx = i \int_0^2 y^2 dy + \int_0^1 x^2 + 4 dx = i \frac{8}{3} + 4 + \frac{1}{3}.$$

$|z|^2$ is not holomorphic, so we should not be too surprised that these integrals are different.

Problem 5. (a) We observe that $e^z + \sin z$ is holomorphic inside the circle $\tilde{C}$ given by $|z-2|=3$. Since 0 is inside this circle, we may use Cauchy's integral formula to see that

$$\int_{\tilde{C}} \frac{e^z + \sin z}{z} dz = 2\pi i(e^0 + \sin 0) = 2\pi i.$$

(b) We start by observing that $\frac{e^z + \sin z}{z}$ is holomorphic inside the circle C $|z-3|=2$ since the only pole (from $\frac{1}{z}$) is outside of this circle. So, Cauchy's theorem gives that

$$\int_C \frac{e^z + \sin z}{z} dz = 0.$$

(c) Here, the integrand has a pole at $z_0 = -i/e$ inside the unit circle since $|z_0| < 1$. So, Cauchy's integral formula gives that

$$(-i/e)^2 e^{-i/e} = \int_{-C'} \frac{z^2 e^z}{e(z - (-i/e))} dz = - \int_{C'} \frac{z^2 e^z}{ez + i} dz.$$

So, the desired integral is equal to $\frac{1}{e^2} e^{-i/e}$.

Problem 6. (*Fundamental theorem of calculus*)¹ Let f have an antiderivative on Ω and $\gamma : [0, 1] \rightarrow \Omega$ any contour in Ω beginning at z_1 and ending at z_2 . Now define functions $u, v : [0, 1] \rightarrow \mathbb{R}$ and $g : [0, 1] \rightarrow \Omega$ by

$$F(\gamma(t)) = g(t) = u(t) + iv(t)$$

The chain rule implies that

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t) = g'(t) = u'(t) + iv'(t).$$

¹This is essentially the same as the fundamental theorem of line integrals for conservative vector fields.

So:

$$\begin{aligned}
\int_{\gamma} F'(z) dz &= \int_0^1 F'(\gamma(t))\gamma'(t) dt = \int_0^1 g'(t) dt \\
&= \int_0^1 u'(t) dt + i \int_0^1 v'(t) dt \\
&= u(1) - u(0) + i(v(1) - v(0)) \\
&= F(\gamma(1)) - F(\gamma(0)) \\
&= F(z_2) - F(z_1),
\end{aligned}$$

where we have used the real fundamental theorem of calculus in the third line. Now, we may evaluate the integral, noting that $e^0 = 1$ and $e^{i\pi} = -1$:

$$\int_{\gamma} e^z + \sin z dz = e^1 - e^{-1} - (\cos 1 - \cos 0) = e - \frac{1}{e} - \cos 1 + 1.$$

Problem 7. (*Primer for the residue theorem*)

(a) We may parameterize $\gamma(t) = e^{it}$, $t \in [0, 2\pi)$. Then $\gamma'(t) = ie^{it}$, and

$$\int_{\gamma} z^n dz = \int_0^{2\pi} e^{itn} (ie^{it}) dt = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1. \end{cases}$$

(b) Integrals are linear, so we may use part (a) to write

$$\oint_{\gamma} f(z) dz = \oint_{\gamma} g(z) dz + \oint_{\gamma} \frac{a_1}{z} dz + \oint_{\gamma} \frac{a_2}{z^2} dz + \cdots + \oint_{\gamma} \frac{a_n}{z^n} dz = 2\pi i a_1,$$

where the integral of $g(z)$ is 0 by Cauchy's theorem.